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A Simple General Treatment of Flavor Oscillations

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Abstract

A unique description avoiding confusion is presented for all flavor-oscillation experiments in which particles of a definite flavor are emitted from a localized source. The probability for finding a particle with the wrong flavor must vanish at the position of the source for all times. This condition requires flavor-time and flavor-energy factorizations which determine uniquely the flavor mixture observed at a detector in the oscillation region; i.e. where the overlaps between the wave packets for different mass eigenstates differ negligibly from 100%. The translation of a "gedanken" experiments calculations (where measurement is perform in time) is done using the *group* velocity. Energy-momentum (frequency-wave number) and space-time descriptions are complementary, equally valid and give the same results. The two identical phase shifts obtained describe the same physics; adding them together to get a factor of two is double counting.

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I. INTRODUCTION

Flavor oscillations are observed when a source creates a particle which is a mixture of two or more mass eigenstates, and a different mixture is observed in a detector. Such oscillations have been observed in the neutral kaon and B-meson systems. In neutrino experiments it is still unclear whether the eigenstates indeed have different masses and whether oscillations can be observed. Considerable confusion has arisen in the description of such experiments in quantum mechanics [1,2], with questions arising about time dependence and production reactions [3], and defining precisely what exactly is observed in an experiment [4]. Many calculations describe "gedanken" experiments and require some recipe for applying the results to a real experiment [5].

We resolve this confusion by noting and applying one simple general feature of all practical experiments. The size of the source is small in comparison with the oscillation wave length to be measured, and a unique well-defined flavor mixture is emitted by the source; e.g. electron neutrinos in a neutrino oscillation experiment. The particles emitted from the source must therefore be described by a wave packet which satisfies a simple general boundary condition: the probability amplitude for finding a particle having the wrong flavor at the source must vanish at all times.

This boundary condition requires factorization of the flavor and time dependence at the position of the source. Since the energy dependence is the Fourier transform of the time dependence, this factorization also implies that the flavor dependence of the wave packet is independent of energy at the position of the source. In a realistic oscillation experiment the phase is important when the oscillation length is of the same order as the distance between the source and the detector. In that case this flavor-energy factorization holds over the entire distance between the source and detector. The boundary condition then determines the relative phase of components in the wave function with different mass having the same energy and different momenta. Thus any flavor oscillations observed as a function of the distance between the source and detector are described by considering only the interference

between a given set of states having the same energy. All questions of coherence, relative phases of components in the wave function with different energies and possible entanglements with other degrees of freedom are thus avoided.

Many formulations describe flavor oscillations in time produced by interference between states with equal momenta and different energies. These "gedanken" experiments have flavor oscillations in time over all space including the source. We show rigorously that the ratio of the wave length of the real spatial oscillation to the period of the gedanken time oscillation is just the group velocity of the wave packet.

II. UNIVERSAL BOUNDARY CONDITION

We now show how the results of a flavor oscillation experiment are completely determined by the propagation dynamics and the boundary condition that the probability of observing a particle of the wrong flavor at the position of the source at any time must vanish. We choose for example a neutrino oscillation experiment with a source of neutrinos of a given flavor, say electron neutrinos. The dimensions of the source are sufficiently small in comparison with the distance to the detector so that it can be considered a point source at $\vec{x} = 0$. The neutrino wave function for this experiment may be a very complicated wave packet, but a sufficient condition for our analysis is to require it to describe a pure ν_e source at $\vec{x} = 0$; i.e. the probability of finding a ν_μ or ν_τ at $\vec{x} = 0$ is zero.

We first consider propagation in free space, where the masses and momenta $\vec{p}_i$ satisfy the usual condition

$$\vec{p}_i{}^2 = E^2 - m_i^2 \tag{2.1}$$

We expand the neutrino wave function in energy eigenstates

$$\psi = \int g(E) dE e^{-iEt} \cdot \sum_{i=1}^{3} c_i e^{i\vec{p}_i \cdot \vec{x}} \left|\nu_i\right\rangle \tag{2.2}$$

where $\left|\nu_i\right\rangle$ denote the three neutrino mass eigenstates and the coefficients c_i are energy-independent. Each energy eigenstate has three terms, one for each mass eigenstate. In order

to avoid spurious flavor oscillations at the source the particular linear combination of these three terms required to describe this experiment must be a pure ν_e state at $\vec{x} = 0$ for each individual energy component. Thus the coefficients c_i satisfy the conditions

$$\sum_{i=1}^{3} c_i \langle \nu_i | \nu_\mu \rangle = \sum_{i=1}^{3} c_i \langle \nu_i | \nu_\tau \rangle = 0 \tag{2.3}$$

The momentum of each of the three components is determined by the energy and the neutrino masses. The propagation of this energy eigenstate, the relative phases of its three mass components and its flavor mixture at the detector are completely determined by the energy-momentum kinematics for the three mass eigenstates.

The exact form of the energy wave packet described by the function $g(E)$ is irrelevant at this stage. The components with different energies may be coherent or incoherent, and they may be "entangled" with other degrees of freedom of the system. For example, for the case where a neutrino is produced together with an electron in a weak decay the function $g(E)$ can also be a function $g(\vec{p}_e, E)$ of the electron momentum as well as the neutrino energy. The neutrino degrees of freedom observed at the detector will then be described by a density matrix after the electron degrees of freedom have been properly integrated out, taking into account any measurements on the electron. However, none of these considerations can introduce a neutrino of the wrong flavor at the position of the source.

Since the momenta $\vec{p}_i$ are energy-dependent the factorization does not hold at finite values of $\vec{x}$. At very large values of $\vec{x}$ the wave packet must separate into individual wave packets with different masses traveling with different velocities [6,1]. However, for the conditions of a realistic oscillation experiment this separation has barely begun and the overlap of the wave packets with different masses is essentially 100%. Under these conditions the flavor-energy factorization introduced at the source is still an excellent approximation at the detector.

The flavor mixture at the detector given by substituting the detector coordinate into Eq. (2.2) can be shown to be the same for all the energy eigenstates except for completely negligible small differences. For example, for the case of two neutrinos with energy E and mass eigenstates m_1 and m_2 the relative phase of the two neutrino waves at a distance x is:

$$\delta\phi(x) = (p_1 - p_2) \cdot x = \frac{(p_1^2 - p_2^2)}{(p_1 + p_2)} \cdot x = \frac{\Delta m^2}{(p_1 + p_2)} \cdot x \tag{2.4}$$

where $\Delta m^2 \equiv m_2^2 - m_1^2$. Since the neutrino mass difference is very small compared to all neutrino momenta and energies, we use $|m_2 - m_1| \ll p \equiv (1/2)(p_1 + p_2)$. Thus we can rewrite eq. (2.4) keeping terms only of first order in $m_2 - m_1$

$$\delta\phi(x) = \frac{\Delta m^2}{2p} \cdot x = -\left(\frac{\partial p}{\partial(m^2)}\right)_E \Delta m^2 \cdot x \tag{2.5}$$

where the standard relativistic energy-momentum relation (2.1) gives the change in energy or momentum with mass when the other is fixed,

$$\left(\frac{2E\partial E}{\partial(m^2)}\right)_p = -\left(\frac{2p\partial p}{\partial(m^2)}\right)_E = 1 \tag{2.6}$$

Thus we have a complete solution to the oscillation problem and can give the neutrino flavor as a function of the distance to the detector by examining the behavior of a single energy eigenstate. The flavor-energy factorization enables the result to be obtained without considering any interference effects between different energy eigenstates. The only information needed to predict the neutrino oscillations is the behavior of a linear combination of the three mass eigenstates having the same energy and different momenta. All effects of interference or relative phase between components of the wave function with different energies are time dependent and are required to vanish at the source, where the flavor is time independent. This time independence also holds at the detector as long as there is significant overlap between the wave packets for different mass states. The conditions for validity of this overlap condition are discussed below.

Neutrino states with the same energy and different momenta are relevant rather than vice versa because the measurement is in space, not time, and flavor-time factorization holds in a definite region in space.

III. RELATION BETWEEN REAL AND GEDANKEN EXPERIMENTS

We now derive the relation between our result (2.4) which comes from interference between states with the same energy and different momenta and the standard treatments using

states with the same momentum and different energies [7]. For the case of two neutrinos with momentum p and mass eigenstates m_1 and m_2 the relative phase of the two neutrino waves at a time t is:

$$\delta\phi(t) = (E_2 - E_1) \cdot t = \left(\frac{\partial E}{\partial(m^2)}\right)_p \cdot \Delta m^2 \cdot t = -\left(\frac{\partial p}{\partial(m^2)}\right)_E \Delta m^2 \cdot \frac{p}{E} \cdot t \tag{3.1}$$

where we have substituted eq. (2.6). This is equal to the result (2.5) if we make the commonly used substitution

$$x = \frac{p}{E} \cdot t = vt \tag{3.2}$$

This is now easily generalized to include cases where external fields can modify the relation (2.1), but where the mass eigenstates are not mixed. The extension to propagation in a medium which mixes mass eigenstates e.g. by the MSW effect [8] is in principle the same, but more complicated in practice and not considered here. The relation between energy, momentum and mass is described by an arbitrary dispersion relation

$$f(E, p, m^2) = 0 \tag{3.3}$$

where the function f can also be a slowly varying function of the distance x. In that case, the momentum p for fixed E is also a slowly varying function of x. We take this into account by expressing Eq. (2.5) as a differential equation, and defining the velocity v by the conventional expression for the group velocity,

$$\frac{\partial^2\phi(x)}{\partial x \partial(m^2)} = -\left(\frac{\partial p}{\partial(m^2)}\right)_E = \frac{1}{v} \cdot \left(\frac{\partial E}{\partial(m^2)}\right)_p , \qquad v \equiv \left(\frac{\partial E}{\partial p}\right)_{(m^2)} \tag{3.4}$$

Treatments describing real experiments measuring distances and "gedanken" experiments measuring time are seen to be rigorously equivalent if the group velocity (3.4) relates the two results. Note that the group velocity and not the phase velocity enters into this relation.

A simple instructive nontrivial example is a toy model dispersion relation qualitatively similar to that for a weak gravitational field [9,10]. A perturbation described by a parameter ϵ is seen to produce effects of opposite sign on the oscillation wave length in space and the period in time

$$(1-\epsilon)E^2 - (1+\epsilon)\vec{p}^2 = m^2 \ , \qquad v = \frac{(1+\epsilon)p}{(1-\epsilon)E} \tag{3.5}$$

$$\delta\phi(x) = \frac{\Delta m^2}{2(1+\epsilon)p} \cdot x \ , \qquad \delta\phi(t) = \frac{\Delta m^2}{2(1-\epsilon)E} \cdot t = \frac{\Delta m^2}{2(1+\epsilon)p} \cdot vt \tag{3.6}$$

Errors are avoided by the use of the correct group velocity [10]. More realistic examples will be given elsewhere [11].

IV. DESCRIPTION IN TERMS OF TIME BEHAVIOR

It is instructive to describe the same physics in terms of the time behavior of the wave function. The specific form of the wave packet given by the function $g(E)$ in Eq. (2.2) describes the Fourier transform of the time behavior as seen at $\vec{x} = 0$. This time behavior changes as the packet moves from the source to the detector. The components corresponding to the different mass eigenstates move with different velocities between the source and the detector. For the case where the wave packets have moved a distance x the centers of the wave packets will have separated by a distance

$$\delta x = \frac{(v_1 - v_2)}{v} \cdot x \approx \frac{(p_1 - p_2)}{p} \cdot x = \frac{\Delta m^2}{2p^2} \cdot x \tag{4.1}$$

where v_1, v_2 and v denote the individual velocities of the two wave packets and an average velocity, and we have assumed that $m_i^2 = E_i^2 - p_i^2 \ll p_i^2$. Here it is clearly the group velocity and not the phase velocity which is relevant, since it is the group velocity which determines the separation between the wave packets. This separation between the wave packet centers produces a phase displacement between the waves at the detector which is seen to give exactly the same phase shift as Eq. (2.4). We see here simply another description of the same physics, using the complementarity of energy-momentum and space-time formulations. They are two ways of getting the same answer, not two different effects that must be added.

The same complementarity is seen in the interference between two classical wave packets moving with slightly different velocities. Even without using the quantum mechanical relations with energy and momentum there are two possible descriptions, one using space and

time variables and one using frequency and wave length. The two descriptions are Fourier transforms of one another and give the same result. Adding the two results is double counting.

Such double counting can arise from noting that the particles traveling with different velocities arrive at the detector at different times [12] and rewriting Eq. (3.1) to obtain an erroneous factor of 2 which has been extensively discussed [4,13–15].

$$x = v_2 t_2 = \frac{p}{E_2} \cdot t_2 = v_1 t_1 = \frac{p}{E_1} \cdot t_1 \ , \qquad \delta\phi(t) = (E_2 t_2 - E_1 t_1) = \frac{\Delta m^2}{p} \cdot x \tag{4.2}$$

The error arises because it is the centers of the two wave packets that arrive at different times, not the detected particle.

Eventually the wave packet separates into distinct packets, one for each mass, moving with different velocities. The separation destroys the flavor-energy and flavor-time factorizations and introduces a time dependence in the flavor observed at a given large distance. When the wave packets for different masses no longer overlap there is no longer any coherence and there are no further oscillations [6]. The result (2.4) applies for the case where the separation (4.1) is small compared to the length in space of the wave packet; i.e. when the eventual separation of the wave packets has barely begun and can be neglected.

V. FUZZINESS IN TIME

The oscillations can be described either in space or in time. But the distance between the source and the detector is known in a realistic experiment to much higher accuracy then the time interval. Thus the interval between the two events of creation and detection has a sharp distance and a fuzzy time in the laboratory system. The fuzziness of the time is an essential feature of the experiment and can be seen explicitly as follows: Let the center of the wave packet leave the source at $\vec{x} = 0$ at the time $t = 0$. Then the particle will be observed at the detector at point x at a time $t \pm \delta t$, where t is the time at which the center of the wave packet arrives at x and δt is the fluctuation in the arrival time associated with

the length of the wave packet. The proper time interval τ between emission and detection is given by

$$\tau^2 = (t \pm \delta t)^2 - x^2 = \frac{m^2}{E^2} t^2 + (\delta t)^2 \pm 2t\delta t \tag{5.1}$$

The length of the wave packet must be sufficiently large to contain a large number of cycles in order to define a phase, $E \cdot \delta t \equiv N \gg 1$. For an oscillation of the order of one cycle to be observed at the time t between two waves differing in energy by δE or at the point x between two waves differing in momentum by δp, $\delta E \cdot t \approx 1 \approx \delta p \cdot x$. Thus

$$\tau^2 \approx \frac{m^2}{E^2 \delta E^2} + \frac{N^2}{E^2} \pm \frac{2N}{E\delta E} = \frac{1}{E\delta E} \cdot \left(\frac{m^2}{E\delta E} + \frac{N^2 \delta E}{E} \pm 2N \right) \tag{5.2}$$

The uncertainty in the proper time interval due to the finite length of the wave packet is seen to be much greater than the value of the proper time interval.

The proper time interval between the two events is always fuzzy. In the laboratory system distance is sharp and time is fuzzy. A Lorentz transformation to a different frame necessarily mixes distance and time and makes both fuzzy in a complicated manner. For this reason one must be careful in interpreting any results obtained in other frames than the laboratory system.

The waves describing the propagation of different mass eigenstates can be coherent only if the time interval between creation and detection is not precisely determined and subject to quantum-mechanical fluctuations. Thus the wave packet created at the source must have a sufficient length in time (coherence length) to prevent the determination of its velocity with a precision needed to identify the mass eigenstate. The small dimensions of the source introduce a momentum uncertainty essential for the coherence of the waves of different mass eigenstates. The wave packet describing the experiment must necessarily contain components from different mass eigenstates with the same energy and different momenta.

Conventional experiments measure distances to a precision with an error tiny in comparison with the oscillation wave length to be measured. This is easily achieved in the laboratory. In a "gedanken" experiment where oscillations in time are measured, the experimental apparatus must measure times to a precision with an error tiny in comparison with

the oscillation period to be measured. One might envision an experiment which measures the time the oscillating particle is created by observing another particle emitted at the same time; e.g. an electron emitted in a beta decay together with the neutrino whose oscillation is observed. But if both the time and position of the created particle are measured with sufficient precision a very sharp wave packet is created and the mass eigenstates moving with different velocities quickly separate and there is no coherence and no oscillation.

In reality, when both x and t are measured there are fluctuations in their values. Using $v = x/t$ the fluctuations in x and t must be large enough to make the velocity fuzzy. We write $v = v_G \pm \delta v$, where v_G is the group velocity and δv denotes the variation in v due to the uncertainty in the wave packet. Then, in order to have oscillation we need $\delta v \gg v_{m_1} - v_{m_2}$ where v_{m_i} is the velocity of the i's mass eigenstates. This is the case in a real experiment. Typical values are [16] $E = O(10\ MeV)$; $x = O(10^2\ m)$; $t = O(10^{-6}\ sec)$ and the relevant masses that can be probed are $\Delta m^2 = O(1\ eV^2)$. Then, $v_{m_1} - v_{m_2} = O(10^{-12})$. Since $\delta v \approx dx/x + dt/t$ we see that the accuracy needed to measure the separate velocities are $dx = O(10^{-10}\ m)$ and $dt = O(10^{-18}\ sec)$, far from the ability of present technology. This calculation can also be performed for all terrestrial experiments, finding that the present technology is always such that oscillations can be seen.

VI. CONCLUSIONS

A unique prescription has been given for the relative phases of the contributions from different mass eigenstates to a flavor oscillation experiment with a localized source having a well defined flavor mixture. The boundary condition that the probability of observing a particle of the wrong flavor at the source position must vanish for all times requires a factorization in flavor and energy of the wave function at the position of the source. This uniquely determines the wave length of the oscillations observed at the detector as long as the overlap between wave packets for different mass eigenstates is maintained at the position of the detector.

Whether this wave-packet overlap is sufficiently close to 100% at the detector depends upon other parameters in the experiment which determine the detailed time behavior of the wave packet. If this overlap is appreciable but no longer nearly complete, the time behavior of the flavor mixture observed at the detector can be extremely complicated with leading and trailing edges of the wave packet being pure mass eigenstates and the intermediate region having a changing flavor mixture depending upon the relative magnitudes of the contributing mass eigenstates as well as the relative phases.

A unique prescription has been given for interpreting results of calculations for "gedanken" experiments which measure oscillations in time for wave packets having the same momentum and different energies. The period of oscillation in time is related to the wave length of oscillation in space by the group velocity of the waves.

Results are simple in the laboratory system where the positions of the source and detector are sharp in comparison with all other relevant distances, and times and proper times must be fuzzy to enable coherent oscillations to be observed.

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REFERENCES

[1] B. Kayser, Phys. Rev. D 24 (1981) 110.

[2] H.J. Lipkin, Neutrino Oscillations and MSW for Pedestrians, Lecture Notes (unpublished).

[3] T. Goldman, hep-ph/9604357.

[4] H.J. Lipkin, Phys. Lett. B 348 (1995) 604.

[5] M.M. Nieto, hep-ph/9509370, Hyperfine Interactions (in press).

[6] S. Nussinov, Phys. Lett. B 63 (1976) 201.

[7] For a review see *e.g.* S.M. Bilenky and S.T. Petcov, Rev. Mod. Phys. 59 (1987) 671; J.N. Bahcall, *Neutrino Astrophysics* (Cambrige University Press, 1989); C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics* (Harwood academic publishers, 1993).

[8] For a review see *e.g.* T.K. Kuo and J. Pantaleone, Rev. Mod. Phys. 61 (1989) 937.

[9] L. Stodolsky, Gen. Rel. Grav. 11 (1979) 391.

[10] D.V. Ahluwalia and C. Burgard, gr-qc/9603008, Gen. Rel. Grav. (in press).

[11] Y. Grossman and H.J. Lipkin, in preparation.

[12] Y.N. Srivastava, A. Widom and E. Sassaroli, hep-ph/9511294.

[13] B. Kayser and L. Stodolsky, Phys. Lett. B 359 (1995) 343.

[14] J. Lowe, B. Bassalleck, H. Burkhardt, A. Rusek, G.J. Stephenson Jr. and T. Goldman, hep-ph/9605234.

[15] B. Ancochea, A. Bramon, R. Munũz-Tapia and M. Nowakowski, hep-ph/9605454.

[16] See *e.g.* Proceedings of the 16th International Conference on Neutrino Physics and Astrophysics, Eilat, Israel, Nucl. Phys. B (Proc. Suppl.) 38 (1995),